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Revisiting Zitterbewegung

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Abstract The Dirac wave equation for the electron soon lead to the recognition of the Zitterbewegung. This was studied both by Schrodinger and Dirac. Later there were further elegant and sometimes dissenting insights, from different authors. We briefly review some of these developments. However in more recent times with dark energy and noncommutative spacetime coming to centre stage, the earlier studies of Zitterbewegung become questionable.

Keywords Zitterbewegung · Schrodinger · Dirac · Noncommutative spacetime

1 Introduction

The phenomenon of Zitterbewegung has been encountered from the very early days of relativistic Quantum Mechanics. The term itself was coined by Schrodinger. It shows up in the Dirac theory of the electron [1]. Early researchers like Dirac himself and Schrodinger studied it. Both of them realized that this effect, in which the Dirac electron appears vibrating with the velocity of light implies the breakdown of the concept of the measurement of the velocity instantaneously: The electron vibrates rapidly within the Compton scale. Our physics starts with averages over the Compton scale, when we return to sub luminal velocities. In other words the electron has this high frequency oscillatory motion superimposed over its normal average motion. If we try to measure the position or velocity of the electron to better than the Compton scale, indeed we end up with the creation of electron positron pairs. This shows up in the complex character of the Dirac coordinates of the electron, or

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equivalently the non Hermitian character of the corresponding operators. So these operators have been considered to be representing a many particle observable. What this means in a one particle interpretation is that the trajectory of the electron cannot be defined by a sharp curve, but rather by a fudged out or fuzzy tube, whose radius is of the order of the Compton wavelength. Let us see this in greater detail [2].

The Hamiltonian for the free electron-positron system according to Dirac is

$$H = c\vec{\alpha} \cdot \vec{p} + mc^2\beta, \tag{1}$$

where $\vec{\alpha}$ and β are matrices which satisfy the Dirac algebra

$$\begin{aligned} \{\alpha_\imath, \alpha_j\} &= 2\delta_{\imath j} \quad (\imath, j = 1, 2, 3), \\ \{\alpha_1, \beta\} &= 0, \quad \beta^2 = I. \end{aligned} \tag{2}$$

The momentum vector $\vec{p}$ and the coordinate vector $\vec{x}$ are taken to satisfy

$$\begin{aligned} [x_\imath, x_j] &= 0 = [p_\imath, p_j], \\ [x_\imath, p_j] &= \imath\hbar\delta_{\imath j} I, \end{aligned} \tag{3}$$

and to commute with $\vec{\alpha}$ and β.

In the Heisenberg picture, all these relations hold at any time, and the time derivative of any one of these operators which do not depend explicitly on time, say A, is given by

$$\frac{dA}{dt} = \imath[H, A]/\hbar. \tag{4}$$

Thus

$$\frac{d\vec{p}}{dt} = \vec{0}, \quad \frac{dH}{dt} = 0, \tag{5}$$

while

$$\frac{d\vec{x}}{dt} = c\vec{\alpha} \tag{6}$$

and

$$-\imath\hbar = \frac{d\vec{\alpha}}{dt} = [H, \vec{\alpha}] = -\{H, \vec{\alpha}\} + 2H\vec{\alpha} = -2c\vec{p} + 2H\vec{\alpha}. \tag{7}$$

This can be written as

$$-\imath\hbar\frac{d\vec{\alpha}}{dt} = 2H\vec{\eta}, \tag{8}$$

with

$$\vec{\eta} = \vec{\alpha} - cH^{-1}\vec{p}. \tag{9}$$

We have

$$-\imath\hbar\frac{d\vec{\eta}}{dt} = -\imath\hbar\frac{d\vec{\alpha}}{dt} = 2H\vec{\eta} \tag{10}$$

so that

$$\vec{n}(t) = e^{2\imath Ht/\hbar_{\vec{\eta}_0}}. \tag{11}$$

Here $\vec{\eta}_0$ is a constant operator:

$$\vec{\eta}_0 = \vec{\eta}(0) = \vec{\alpha}(0) - cH^{-1}\vec{p}. \tag{12}$$

We can easily check that

$$\{H, \vec{\eta}\} = 0 = \{H.\vec{\eta}_0\} \tag{13}$$

so that we can also write, from (11),

$$\vec{\eta}(t) = \vec{\eta}_0 e^{-2\imath Ht/\hbar}. \tag{14}$$

Combining (7), (9) and (14), we obtain

$$\frac{d\vec{x}}{dt} = c\vec{\alpha} = c^2 H^{-1}\vec{p} + c\vec{\eta}_0 e^{-2\imath Ht/\hbar} \tag{15}$$

which can be integrated again to get

$$\vec{x}(t) = \vec{a} + c^2 H^{-1}\vec{p}t + \frac{1}{2}\imath\hbar c\vec{\eta}_0 H^{-1} e^{-2\imath Ht/\hbar}, \tag{16}$$

with $\vec{a}$ a constant (operator) of integration

$$\vec{a} = \vec{x}(0) - \frac{1}{2}\imath\hbar c\vec{\alpha}(0)H^{-1} + \frac{1}{2}\imath\hbar c^2 H^{-2}\vec{p}. \tag{17}$$

Now (16) can be written as

$$\vec{x}(t) = \vec{x}_A(t) + \vec{\xi}(t) \tag{18}$$

with

$$\vec{x}_A(t) = \vec{a} + c^2 H^{-1}\vec{p}t, \tag{19}$$

the form we might expect for the "position" operator of a relativistic classical point mass. The remaining contribution to $\vec{x}$ is

$$\vec{\xi}(t) = \frac{1}{2}\imath\hbar c\vec{\eta}_0 H^{-1} e^{-2\imath Ht/\hbar} = \frac{1}{2}\imath\hbar c\vec{\eta} H^{-1}, \tag{20}$$

which describes a microscopic, high-frequency Zitterbewegung superimposed on the microscopic motion (19) associated with $\vec{x}_A$. The Zitterbewegung has a characteristic amplitude $\hbar/2mc$, half the Compton wavelength of the electron, and a characteristic angular frequency $2mc^2/\hbar$.

Nevertheless Zitterbewegung has held the attention of a number of researchers over the decades and we will now revisit these interpretations and finally make some comments.

2 Huang's Approach

In 1952 K. Huang made an elegant and precise study of Zitterbewegung [3]. As this work has been all but forgotten, we reproduce some important parts of it. We begin by observing that the general Dirac wave packet may be written in a momentum space expansion of the form

$$\psi(x,t) = h^{-1}\int [C^{+}(p)e^{-\imath wt} + C^{-}(p)e^{\imath wt}]\exp(\imath p\cdot r/\hbar)d^3p, \tag{21}$$

where $w = \epsilon/\hbar, \epsilon = +(c^2p^2 + m^2c^4)^{\frac{1}{2}}, C^{+}(p)$ is a linear combination of "spin up" and "spin down" amplitudes of the free particle Dirac waves belonging to the momentum p with positive energy, and $C^{-}(p)$ for negative energy. That is,

$$\begin{aligned} C^{+}(p) &= a_1u_1 + a_2u_2, \\ C^{-}(p) &= a_3u_3 + a_4u_4, \end{aligned} \tag{22}$$

where the coefficients a_k are complex functions of p, and

$$\begin{aligned} u_1 &= \begin{bmatrix} 1 \\ 0 \\ kp_3 \\ kp_+ \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ kp_- \\ -l[_3 \end{bmatrix}, \\ u_3 &= \begin{bmatrix} -kp_3 \\ -kp_+ \\ 1 \\ 0 \end{bmatrix}, \quad u_4 = \begin{bmatrix} -kp_- \\ kp_3 \\ 0 \\ 1 \end{bmatrix}, \end{aligned} \tag{23}$$

with

$$k = \frac{c}{\epsilon + mc^2}, \quad p\pm = p_1 \pm \imath p_2. \tag{24}$$

The wave packet, (22), is normalized provided we require that

$$\int [C^{+*}(p)C^{+}(p) + C^{-*}(p)C^{-}(p)]d^2p = 1.$$

Hence,

$$\begin{aligned} \langle t\rangle = c\langle\alpha\rangle &= c\int (C^{+*}\alpha C^{+} + C^{-*}\alpha C^{-})d^3p \\ &\quad + c\int [(C^{+*}\alpha C^{-})e^{2\imath wt} + (C^{-*}\alpha C^{+})e^{-2\imath wt}]d^2p. \end{aligned}$$

the arguments of $C^{\pm}$ being understood. Since α is a Hermitian operator, we have

$$\begin{aligned} \langle t\rangle &= c\int (C^{+*}\alpha C^{+} + C^{-*}\alpha C^{-})d^2p \\ &\quad + 2c\int \mathrm{Re}\{(C^{+*}\alpha C^{-})e^{2\imath wt}\}d^2p, \\ \langle t\rangle &= \bar{\upsilon} + 2c\int K(p)\sin(2wt + \phi(p))d^2p, \end{aligned} \tag{25}$$

where

$$\bar{v} = c\int \frac{p}{mc}\left[1+\left(\frac{p}{mc}\right)^2\right]^{-\frac{1}{2}}(C^{+*}C^{+} - C^{-*}C^{-})d^2p,$$
$$K(p) = |C^{-*}\alpha C^{+}|,$$
$$\phi(p) = \tan^{-1}\{\mathrm{Re}(C^{-*}\alpha C^{+})/\mathrm{Im}(C^{-*}\alpha C^{+})\}. \tag{26}$$

Hence

$$\langle r\rangle = r^0 + \bar{v}t - \lambda\int K(p)[1+(p/mc)^2]^{-\frac{1}{2}} \times \cos(2wt+\phi(p))d^2p, \tag{27}$$

where $\lambda = \hbar/mc$ is the Compton wavelength (divided by 2π) of the electron.
The expectation value for the position is seen to be composed of two parts; a time-independent part, which describes the average motion of the electron, and a time-dependent part, which involves interference terms of the type $C^{+*}\alpha C^{-}$. It arises from the interference between the positive and negative energy states of the electron, and represents the Zitterbewegung of the electron. This is as in (17).

For spin along the z axis, Huang finally deduces:

$$\langle x_1\rangle_\psi = -\lambda\sin(2wt+\psi),$$
$$\langle x_2\rangle_\psi = -\lambda\cos(2wt+\psi). \tag{28}$$

We can deduce that while this represents a circulatory motion, the centre itself is at rest, in this case. Further, the x and y components of $\vec{r}\times\vec{r}^{\circ}$ vanish on the average but the z component is proportional to the magnetic moment of the electron. To sum up:

The detailed motion of a free Dirac electron was investigated by examining the expectation values of the position $\vec{r}$ and of $\vec{r}\times\dot{\vec{r}}$ in a wave packet. It was shown by Huang that the well-known Zitterbewegung may be looked upon as a circular motion about the direction of the electron spin, with a radius equal to the Compton wavelength (divided by 2π) of the electron. Further the intrinsic spin of the electron may be looked upon as the "orbital angular momentum" of this motion. The current produced by the Zitterbewegung is seen to give rise to the intrinsic magnetic moment of the electron.

3 The Approaches of Barut and Hestenes

Schrodinger's proposed "microscopic momentum" vector of the Zitterbewegung was rejected by Barut in favor of a "relative momentum" vector, with the value $\vec{P} = c\vec{d}$ in the rest frame of the center of mass. His oscillatory "microscopic coordinate" vector was retained in the rest frame, taking the form $\vec{Q} = \imath(\hbar/2mc)\beta\vec{d}$, and the Zitterbewegung was described in this frame in terms of $\vec{P}$, $\vec{Q}$, and the Hamiltonian $mc^2\beta$, as a finite three-dimensional harmonic oscillator with a compact phase space. The Lie algebra generated by $\vec{Q}$ and $\vec{P}$ is that of $SO(5)$, and in particular $[Q_\imath, P_j] = -\imath\hbar\delta_{\imath j}\beta$. Barut argued that the simplest possible finite, three-dimensional, isotropic, quantum-mechanical system requires such as $SO(5)$ structure, incorporates a fundamental length, and has harmonic-oscillator dynamics. Dirac's equation was derived as the wave equation appropriate to the description of such a finite quantum system in an arbitrary moving frame of reference, using a dynamical group $SO(3,2)$ which can be extended to $SO(4,2)$. Spin appears here as the orbital angular momentum associated with the internal system, and rest-mass energy appears as the internal energy in the rest frame.

Hestenes [4] on the other hand struck a different note. He observed, "The Zitterbewegung is a local circulatory motion of the electron presumed to be the basis of the electron spin and magnetic moment. A reformulation of the Dirac theory shows that the Zitterbewegung need not be attributed to interference between positive and negative energy states as originally proposed by Schrodinger. Rather, it provides a physical interpretation for the complex phase factor in the Dirac wave function generally. Moreover, it extends to a coherent physical interpretation of the entire Dirac theory, and it implies a Zitterbewegung interpretation for the Schrodinger theory as well."

4 The Author's Interpretation

As can be seen from (15), the motion of the Dirac electron consists of two parts. The first part represents the usual Newtonian motion of a particle. However, the second part is the very high frequency Zitterbewegung motion which is superimposed on the Newtonian motion. The frequency here is of the order of the Compton frequency that is $mc^2/\hbar$. This motion was elegantly studied by Huang as described in Sect. 2. Indeed, as can also be seen from (28), this Zitterbewegung part represents a circulatory motion with velocity c. Indeed the radius of this circuit is of the order of the Compton wavelength λ. We can see that the circulation is given by

$$I = \oint \vec{p} \cdot d\vec{s}, \tag{29}$$

where the circuit has the radius equalling the Compton wavelength $\lambda = h/2mc$. Using the fact obtained from (28) that p is given by mc, the circulation in (29) now becomes

$$I = \frac{h}{2}. \tag{30}$$

Equation (30) shows that the circulation gives the usual spin of the electron. The picture that now emerges is that the spinning electron mimics a vortex which as usual exhibits two motions. The first is the circulatory motion which represents the electron spin, and the second is the motion of the vortex as a whole, given by the Newtonian velocity in (15) [5–7].

Let us now look at all this in a more recent perspective. We note that from the realm of Quantum Mechanics the position coordinate for a Dirac particle in conventional theory is given by

$$x = (c^2 p_1 H^{-1} t) + \frac{\imath}{2} c\hbar(\alpha_1 - c p_1 H^{-1}) H^{-1} \tag{31}$$

an expression that is very similar to (16). Infact as was argued in detail [5] the imaginary parts of both (16) and (31) are the same, being of the order of the Compton wavelength.

It is at this stage that a proper physical interpretation begins to emerge. Dirac himself observed as noted, that to interpret (31) meaningfully, it must be remembered that Quantum Mechanical measurements are really averaged over the Compton scale: Within the scale there are the unphysical Zitterbewegung effects: for a point electron the velocity equals that of light.

Once such a minimum spacetime scale is invoked, then we have a non commutative geometry as shown by Snyder more than fifty years ago [8, 9]:

$$[x, y] = (\imath a^2/\hbar)L_z, [t, x] = (\imath a^2/\hbar c)M_x, \text{etc.}$$
$$[x, p_x] = \imath\hbar[1 + (a/\hbar)^2 p_x^2]. \tag{32}$$

Moreover relations (32) are compatible with Special Relativity. Indeed such minimum spacetime models were studied for several decades, precisely to overcome the divergences encountered in Quantum Field Theory [5, 9–16].

Before proceeding further, it may be remarked that when the square of a, which we will take to be the Compton wavelength (including the Planck scale, which is a special case of the Compton scale for a Planck mass viz., 10^{-5} g m), can be neglected, then we return to point Quantum Theory and the usual commutative geometry.

It is interesting that starting from the Dirac coordinate in (31), we can deduce the non commutative geometry (32), independently. For this we note that the α's in (31) are given by

$$\vec{\alpha} = \begin{bmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{bmatrix},$$

the σ's being the Pauli matrices. We next observe that the first term on the right hand side is the usual Hermitian position. For the second term which contains α, we can easily verify from the commutation relations of the σ's that

$$[x_\imath, x_j] = \beta_{\imath j} \cdot l^2, \tag{33}$$

where l is the Compton scale.

There is another way of looking at this. Let us consider the one dimensional coordinate in (31) or (16) to be complex. We now try to generalize this complex coordinate to three dimensions. Then we encounter a surprise—we end up with not three, but four dimensions,

$$(1, \imath) \to (I, \sigma),$$

where I is the unit 2×2 matrix and σs are the Pauli matrices. We get the special relativistic Lorentz invariant metric at the same time. (In this sense, as noted by Sachs [17], Hamilton who made this generalization would have hit upon Special Relativity, if he had identified the new fourth coordinate with time.)

That is,

$$x + \imath y \to I x_1 + \imath x_2 + j x_3 + k x_4,$$

where $(\imath, j, k)$ now represent the Pauli matrices; and, further,

$$x_1^2 + x_2^2 + x_3^2 - x_4^2$$

is invariant. Before proceeding further, we remark that special relativistic time emerges above from the generalization of the complex one dimensional space coordinate to three dimensions.

While the usual Minkowski four vector transforms as the basis of the four dimensional representation of the Poincare group, the two dimensional representation of the same group, given by the right hand side in terms of Pauli matrices, obeys the quaternionic algebra of the second rank spinors (cf. Refs. [17–19] for details).

To put it briefly, the quarternion number field obeys the group property and this leads to a number system of quadruplets as a minimum extension. In fact one representation of the two dimensional form of the quarternion basis elements is the set of Pauli matrices. Thus a quarternion may be expressed in the form

$$Q = -\imath\sigma_\mu x^\mu = \sigma_0 x^4 - \imath\sigma_1 x^1 - \imath\sigma_2 x^2 - \imath\sigma_3 x^3 = (\sigma_0 x^4 + \imath\vec{\sigma}\cdot\vec{r}).$$

This can also be written as

$$Q = -\imath\begin{pmatrix} \imath x^4 + x^3 & x^1 - \imath x^2 \\ x^1 + \imath x^2 & \imath x^4 - x^3 \end{pmatrix}.$$

As can be seen from the above, there is a one to one correspondence between a Minkowski four-vector and Q. The invariant is now given by $Q\bar{Q}$, where $\bar{Q}$ is the complex conjugate of Q.

However, as is well known, there is a lack of spacetime reflection symmetry in this latter formulation. If we require reflection symmetry also, we have to consider the four dimensional representation,

$$(I,\vec{\sigma}) \to \left[\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\right] \equiv (\Gamma^\mu)$$

(cf. also Ref. [20] for a detailed discussion). The motivation for such a reflection symmetry is that usual laws of physics, like electromagnetism do indeed show the symmetry.

We at once deduce spin and Special Relativity and the geometry (32) in these considerations. This is a transition that has been long overlooked [21]. It must also be mentioned that spin half itself is relational and refers to three dimensions, to a spin network infact [22, 23]. That is, spin half is not meaningful in a single particle Universe.

While a relation like (33) above has been in use recently, in non commutative models, we would like to stress that it has been overlooked that the origin of this non commutativity lies in the original Dirac coordinates. (Conversely, starting with the noncommutative geometry (33), we can argue that the Dirac equation can be obtained (cf. Ref. [7]).)

The above relation shows on comparison with the position-momentum commutator that the coordinate $\vec{x}$ also behaves like a "momentum". This can be seen directly from the Dirac theory itself where we have [24]

$$c\vec{\alpha} = \frac{c^2\vec{p}}{H} - \frac{2\imath}{\hbar}\hat{x}H. \tag{34}$$

In (34), the first term is the usual momentum. The second term is the extra "momentum" $\vec{p}$ due to Zitterbewegung we have already encountered.

Infact we can easily verify from (34) that

$$\vec{p} = \frac{H^2}{\hbar c^2}\hat{x}, \tag{35}$$

where $\hat{x}$ has been defined in (34).

We finally investigate what the angular momentum $\sim \vec{x}\times\vec{p}$ gives—that is, the angular momentum at the Compton scale. We can easily show that

$$(\vec{x}\times\vec{p})_z = \frac{c}{E}(\vec{\alpha}\times\vec{p})_z = \frac{c}{E}(p_2\alpha_1 - p_1\alpha_2), \tag{36}$$

where E is the eigen value of the Hamiltonian operator H. Equation (36) shows that the usual angular momentum but in the context of the minimum Compton scale cut off, leads to the "mysterious" Quantum Mechanical spin. This resembles considerations in the previous section.

In the above considerations, we started with the Dirac equation and deduced the underlying non commutative geometry of spacetime.

In recent years, dark energy or the vacuum Zero Point Field has occupied centre stage. Indeed the author's 1997 work pointed to a dark energy driven accelerating universe with a small cosmological constant and this was dramatically confirmed the very next year (cf. Ref. [5] and references therein for details).

We would first like to point out that a background Zero Point Field of this kind can explain the Quantum Mechanical spin half (as also the anomalous $g = 2$ factor) for an otherwise purely classical electron [6, 25, 26]. The key point here is (cf. Ref. [25]) that the classical angular momentum $\vec{r} \times m\vec{v}$ does not satisfy the Quantum Mechanical commutation rule for the angular momentum $\vec{J}$. However when we introduce the background Zero Point Field (ZPF), the momentum now becomes

$$\vec{J} = \vec{r} \times m\vec{v} + (e/2c)\vec{r} \times (\vec{B} \times \vec{r}) + (e/c)\vec{r} \times \vec{A}^0, \tag{37}$$

where $\vec{A}^0$ is the vector potential associated with the ZPF and $\vec{B}$ is an external magnetic field introduced merely for convenience, and which can be made vanishingly small.

It can be shown that $\vec{J}$ in (37) satisfies the Quantum Mechanical commutation relation for $\vec{J} \times \vec{J}$. At the same time we can deduce from (37)

$$\langle J_z \rangle = -\frac{1}{2}\hbar\omega_0/|\omega_0|. \tag{38}$$

Relation (38) gives the correct Quantum Mechanical results referred to above. From (37) we can also deduce that

$$l = \langle r^2 \rangle^{\frac{1}{2}} = \left(\frac{\hbar}{mc}\right). \tag{39}$$

Equation (39) shows that the mean dimension of the region in which the ZPF fluctuation contributes is of the order of the Compton wavelength of the electron. By relativistic covariance (cf. Ref. [6]), the corresponding time scale is at the Compton scale. As noted, Dirac, in his relativistic theory of the electron encountered the Zitterbewegung effects within the Compton scale [1] and he had to invoke averages over this scale to recover meaningful results. We have also shown earlier, without invoking the ZPF [21] how spin follows, that is we get (38) and (39) using Zitterbewegung.

In any case, the earlier precise characterization of Zitterbewegung assumed that time and space can be precisely characterized at arbitrarily small scales. This assumption does not stand the scrutiny of later work [7, 27].

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